Exact Finite Difference Schemes for Solving Helmholtz Equation at Any Wavenumber: reproducing the results

# Purpose

Reproduce the results of the publication [[1](#YAU01)]. The ultimate purpose being to use this method to our problem: the computation of the Helmholtz equation:

|  |  |  |
| --- | --- | --- |
|  |  | (1) |

For sake of pedagogical purpose, we want to reproduce the new algorithm method for the one dimensional problem and the two dimensional problem exposed in the work of [[1](#YAU01)] and compare it with the ancient scheme. The latter is not explicitly given for what concerns the boundary and especially how to apply the Sommerfeld condition and has been rebuilt thanks to the explanation given in [[2](#Heg10)]. A pedagogical explanation about the general scheme matrix building process may be found in [[3](#LeV07)] and [[4](#Cha10)].

# Protocol

The computation of the Helmholtz equation is done for the one dimensional and the two dimensional problem.

Two different schemes are available for the computation of the interior points: the classical 3 (1 dimension) and 5 (2 dimension) schemes and the new version of this scheme. Three different schemes are available for the computation of the boundary points: Dirichlet boundary, the Sommerfeld condition with classical central difference and the Sommerfeld condition with the new difference scheme.

## The one dimensional problem

The following equations are valid for the one-dimensional problem.

|  |  |  |
| --- | --- | --- |
|  |  | (2) |

The following schemes may be calculated from the second order Taylor series. For a demonstration of each of these, see (ref. appendices).

### Classical Scheme

The classical algorithm is composed of four schemes. These give the coefficients that are given to the elements of the scheme matrix and, in case of Dirichlet constraint, of the vector.

|  |  |
| --- | --- |
| Points type | Scheme |
| Interior |  |
| Dirichlet |  |
| Sommerfeld (right) |  |
| Sommerfeld (left) |  |

### New scheme

The classical algorithm is composed of four schemes. These give the coefficients that are given to the elements of the scheme matrix and, in case of Dirichlet constraint, of the vector.

|  |  |
| --- | --- |
| Points type | Scheme |
| Interior |  |
| Dirichlet |  |
| Sommerfeld (right) |  |
| Sommerfeld (left) |  |

## The two dimensional problem

# Results

We consider being a function sufficiently smooth (i.e. a sufficiently high number of time derivable).

It is possible to define different approximations of this function. Also we do not have the effective formulas for the moment we will write them the following.

|  |  |  |
| --- | --- | --- |
| Left approximation | Centered approximation | Right approximation |
|  |  |  |

The preceding notation is for an approximation of the first order. Second and higher approximation may be indicated by an exponent. For instance the 3 order approximation of the function from the right would be indicated by the following: .

# Taylor expansion

## The basic Taylor formula

We will now give the basic formula of a Taylor expansion and other more convenient forms of these formulas that once assemble allow producing schemes.

For our function of a scale variable the Taylor formula sufficiently near a point may be written the following:

|  |  |  |
| --- | --- | --- |
|  |  | (3) |

This may be rewritten the following by letting:

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

## Different variation on the extension

It is possible to derive a wide range of Taylor expansion depending on the point we wish to calculate it. The only thing to care of is that the variable tends to a when h tends to 0.

From (4) we can derive:

|  |  |  |
| --- | --- | --- |
|  |  | (5) |

|  |  |  |
| --- | --- | --- |
|  |  | (6) |

|  |  |  |
| --- | --- | --- |
|  |  | (7) |

These expansions will serve as basis bricks to build computation schemes.

## Remark

1. These estimations depend on
2. These estimations are local. The more h is small the more they will be precise.
3. The function is the truncation error. It will give an idea of how much decimal are significant in the result. Here also if h is sufficiently small, it will increase the degree of precision of the calculation (a trade-off exists between the size of h, the memory handled and the computation time).

# Bibliography

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| [4] | S C Chapra and R P Canale, *Numerical Methods For Engineers*, Sixth Edition ed., Mac Graw Hill, Ed., 2010. |

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